Homogeneous Number System and Reciprocal Symmetric Algebra

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Abstract

We have shown that an isomorphic transformation relates the set $R$ of real numbers extending from $-\infty$ to $+\infty$, to the set $S$ of numbers between arbitrarily chosen numbers $c$ and $-c$. Any set of physical quantities which can be represented by $R$ can also be represented by $S$ and vice versa. $c$ corresponds to $\infty$, and invariance of $c$ under Einstein addition corresponds to invariance of $\infty$ under ordinary addition.

Keywords: Real numbers; Isomorphism; Neutral number; Infinity; Invariant under addition; Einstein law of addition; Reciprocal symmetry

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INTRODUCTION

It is customary to use the familiar arithmetic to represent measurement. In this representation null is represented by 0, quantities too big to measure are represented by $\infty$. This is not the only choice. We could replace 0 by any arbitrarily chosen $\varphi$, and $\infty$ by any arbitrarily chosen $c_\varphi \neq \varphi$. We have to find the arithmetic operations which correctly translate the properties of quantities to be measured. Since all the different representations (depending on the choice of $\varphi$, and $c_\varphi$) represent the same quantity, all the representations must be isomorphically related.

GEOMETRIC REPRESENTATION OF HOMOGENOUS NUMBER SYSTEM

Points on a circle represent number. All numbers have the same status

\[
\Omega = c \tan(\pm \pi / 2)
\]

\[
\Omega \neq \infty \text{ because } -\Omega = +\Omega
\]

\[a = c \tan(\theta / 2) \quad \text{and} \quad a^* = c^2 / a\]

$c$ is the scale number.

Figure 1
**Einsteinian numbers**

Numbers on right hand semi circle are Einsteinian numbers

\[-c \leq u, \ v \leq c\]  

Figure 2

Addition closed in the semi circle (Einstein addition) is

\[-c \leq u \oplus_0 v = \frac{u + v}{1 + u.v/c^2} \leq c\]  

(1)

Invariant under addition (Einstein’s postulate)

\[u \oplus_0 (\pm c)v = \pm c\]  

(2)

Reciprocal Symmetry

\[u \oplus_0 v = (u^*) \oplus_0 (v^*) \text{ where } u^* = c^2 / u\]  

(3)

**Reciprocal set**

Every number \(a\) has its reciprocal \(a^* = c^2 / a\). Numbers on left hand semi circle are de Broglie numbers

\[c \leq |u^*|, |v^*|\]  

(4)

Figure 3

Addition closed in the semi circle (Tachyonic addition) is

\[u^* \oplus_\Omega v^* = \frac{u^*.v^* + c^2}{u^* + v^*} = \frac{u.v + c^2}{u + v}\]  

(5)

Reciprocal Symmetry

\[u \oplus_\Omega v = (u^*) \oplus_\Omega (v^*)\]  

(6)

Upper half semi circle are Quantum numbers. Addition closed in the semi circle

(Quantum addition) is \(\psi \oplus_1 \psi’ = \psi \psi’\)

\[R(\psi) = -\psi\]
Reciprocal Symmetry

\[ \psi \oplus_1 \psi' = \psi \psi' = \psi \oplus_1 (-\psi') \quad (7) \]

Invariant under addition

\[ \psi \oplus_1 (0)^{\pm 1} = 0^\pm = 0 \text{ or } \infty \quad (8) \]

**Shift of Origin**

![Figure 5](image)

In Figure 5 the origin has been shifted to \( \varphi \)

**ISOMORPHISM 1**

**Transformation and Addition**

Upper scale numbers \( U, V \) etc. belong to the set \( R \) of real numbers. Lower scale numbers \( u, v \) etc. belong to the set \( S \) of real numbers between \( c \) and \( -c \) and containing 0 [numbers on right hand semi circle of Figure 2]. We shall see below the isomorphism between \( R \) and \( S \).

Let

\[ U = c \cdot \text{arctanh}(u/c) \text{ or } u/c = \tanh(U/c) \quad (9) \]

Then we can write for \( -\infty \leq U/c \leq \infty \)

\[ U/c = \ln \frac{1+u/c}{\sqrt{1-(u/c)^2}} \text{ with } -1 \leq u/c \leq 1 \quad (10) \]

Using (10) the sum \( W = U + V \) becomes

\[ W = U + V = c \ln \frac{1+u/c}{\sqrt{1-(u/c)^2}} \cdot \frac{1+v/c}{\sqrt{1-(v/c)^2}} = c \ln \frac{1+w/c}{\sqrt{1-(w/c)^2}} \quad (11) \]
Where
\[ w = u \oplus v = \frac{u + v}{1 + u.v/c^2} \quad \text{with} \quad -1 \leq w/c \leq 1 \quad \text{if} \quad -c \leq u, v \leq c \quad (12) \]

(10) shows that \( u = c \) corresponds to \( U = \infty \). \( c \) is invariant under Einstein addition \( u \oplus c = c \) just as \( \infty \) is invariant under ordinary addition \( U + \infty = \infty \)

**Law of Multiplication**

From (10)
\[ nU = n.c \ln \frac{1 + u/c}{\sqrt{1 - (u/c)^2}} = c . \ln \left( \frac{1 + u/c}{\sqrt{1 - (u/c)^2}} \right)^n = c . \ln \frac{1 + (n \otimes u)/c}{\sqrt{1 - ((n \otimes u)/c)^2}} \quad (13) \]

Where
\[ n \otimes u = c \frac{\left(1 + u/c\right)^n - 1}{\left(1 - u/c\right)^n + 1} \quad (14) \]

\( n \otimes u \) of (14) is consistent -- in the sense that it is distributive\(^3\) -- with addition (12).
\[ \{m \otimes u\} \oplus \{m \otimes v\} = m \otimes \{u \oplus v\} \quad \text{and} \]
\[ \{m \otimes u\} \oplus \{n \otimes u\} = (m + n) \otimes u \quad (15) \]

The isomorphism makes it possible to measure any quantity (measurable in terms of real numbers) by using only the set of numbers on the right hand semi circle of fig.1. (14) shows \( \infty \otimes u = c \) corresponding to \( \infty U = \infty \)

**Reciprocal Set**

Every number \( a \) has its reciprocal
\[ a^* = c^2/a \quad (16) \]
on the left hand semi circle of fig. 1. If we can define addition closed in the set \( \{a^*\} \geq c \) [left hand semi circle], the set of numbers will be an alternative representation. Consistency requires reciprocal symmetry
\[ u \oplus v = u^* \oplus v^* \quad (17) \]
**ISOMORPHISM, GENERALIZED NEGATIVE AND GENERALIZED RECIPROCAL**

We shall choose \( \varphi \) to be the neutral number and \( c_\varphi \) to be the boundary value number. We want the set of numbers from \( c_\varphi \) to \( N(c_\varphi) \) including \( \varphi \) to be a complete set of numbers. We shall develop the algebraic operations in this set. If \( \varphi = 0 \) and \( c_0 = c \) our chosen set will be set of permissible Einsteinian velocities. If \( u_0 \) represents a physical quantity in the system in which the neutral number is 0, and \( u_\varphi \) represents the same quantity in the system in which the neutral number is \( \varphi \), the transformation relation between \( u_\varphi \) and \( u_0 \) will be given by

\[
u_\varphi = \frac{u_0 + \varphi}{1 - u_0 \varphi/c^2} \quad \text{and} \quad u_0 = \frac{u_\varphi - \varphi}{1 + u_\varphi \varphi/c^2}
\] (18)

If \( u_\varphi \) represents the value of a quantity in the number system in which neutral number is \( \varphi \) and \( u_\lambda \) represents the same quantity in the system in which neutral number is \( \lambda \), the correspondence between \( u_\varphi \) and \( u_\lambda \) is given by the relation

\[
u_\lambda = \frac{u_\varphi - \varphi + \lambda (1 + u_\varphi \varphi/c^2)}{1 + u_\varphi \varphi/c^2 - (\lambda/c^2)(u_\varphi - \varphi)}
\] (19)

When \( \varphi = 0 \) we shall often drop the subscript and write

\[
u_0 = u \quad \text{and} \quad c_0 = c
\] (20)

Generalized negative \( N(u_\varphi) \) and generalized reciprocal \( R(u_\varphi) \) of \( u_\varphi \) are given by

\[
N(u_\varphi) = \frac{-u_\varphi + \varphi (2 + u_\varphi \varphi/c^2)}{(1 + u_\varphi \varphi/c^2) + (u_\varphi - \varphi) \varphi/c^2}
\] (21)

\[
R(u_\varphi) = \frac{\varphi (u_\varphi - \varphi) (1 + c_\varphi \varphi/c^2)^2 + (c_\varphi - \varphi)^2 (1 + u_\varphi \varphi/c^2)}{(u_\varphi - \varphi) (1 + c_\varphi \varphi/c^2)^2 - (c_\varphi - \varphi)^2 (1 + u_\varphi \varphi/c^2) \varphi/c^2}
\] (22)

There are 4 self image numbers \( \varphi, \ R(\varphi), \ c_\varphi \) and \( N(c_\varphi) \) so that

\[
N(\varphi) = \varphi \quad \text{and} \quad N(R(\varphi)) = R(\varphi)
\] (23)

\[
R(c_\varphi) = c_\varphi \quad \text{and} \quad R(N(c_\varphi)) = N(c_\varphi)
\] (24)

Reflections commute
\[ R(N(a_\varphi)) = NR(a_\varphi) \]  
\hspace{1cm} (25)  

Repeated reflections give the number back  
\[ R^2(a_\varphi) = N^2(a_\varphi) = a_\varphi \]  
\hspace{1cm} (26)  

**Generalized Addition**

\( \oplus_\varphi \) represents the law of addition in the set determined by neutral number \( \varphi \) and boundary value number \( c_\varphi \). The sum \( u_\varphi \oplus_\varphi v_\varphi \) must have the following properties

\[ u_\varphi \oplus_\varphi v_\varphi = v_\varphi \oplus_\varphi u_\varphi \]  
\hspace{1cm} (27)  

\[ u_\varphi \oplus_\varphi \varphi = u_\varphi \]  
\hspace{1cm} (28)  

\[ u_\varphi \oplus_\varphi N(u_\varphi) = \varphi \]  
\hspace{1cm} (29)  

\[ R(u_\varphi) \oplus_\varphi R(v_\varphi) = v_\varphi \oplus_\varphi u_\varphi \]  
\hspace{1cm} (30)  

\[ u_\varphi \oplus_\varphi c_\varphi = c_\varphi \]  
\hspace{1cm} (31)  

**Invariance Theorem:** If (30), (25) and (24) are true, then  
\[ u \oplus_\varphi c_\varphi = c_\varphi \] and  
\[ u \oplus_\varphi N_\varphi(c_\varphi) = N_\varphi(c_\varphi) \]  
\hspace{1cm} (32)  

**Proof:** Let  
\[
\begin{align*}
  u \oplus_\varphi v \\
  R(u) \oplus_\varphi R(v)
\end{align*}
\]  
\hspace{1cm} (33)  

Adding \( N(v) \) and \( NR(v) \) to the upper and lower equations above, we get (hiding sub-script \( \varphi \) )

\[ u = w \oplus N(v) \]  
\hspace{1cm} (34)  

\[ R(u) = w \oplus NR(v) \]  
\hspace{1cm} (35)  

Using (25), (35) becomes  
\[ R(u) = w \oplus RN(v) \]  
\hspace{1cm} (36)  

We chose \( N(v) = c_\varphi \) and using (24), (34) and (35) we get

\[ u = w \oplus c_\varphi = R(u) \]  
\hspace{1cm} (37)  

In (37) \( u = R(u) \). This is possible only if

\[ u = R(u) = c_\varphi \] or \[ u = R(u) = N(c_\varphi) \]  
\hspace{1cm} (38)  

Therefore,
\[
 w \oplus c_\varphi = \begin{cases} 
 c_\varphi & \text{or} \\
 N(c_\varphi) & 
\end{cases}
\]

(39) should be true for all values of \( w \) including \( w = 0 \). Therefore,

\[
 w \oplus c_\varphi = c_\varphi \quad \text{Q. E. D.}
\]

All our requirements are fulfilled if

\[
 w_\varphi = u_\varphi \oplus_\varphi v_\varphi = \frac{w_0 + \varphi}{1 - w_0 \varphi / c^2}
\]

Where

\[
 w_0 = c^2(c_\varphi - \varphi)^2 \frac{(u_\varphi - \varphi)(c^2 + v_\varphi \varphi) + (v_\varphi - \varphi)(c^2 + u_\varphi \varphi)}{(c_\varphi - \varphi)^2(c^2 + u_\varphi \varphi)(c^2 + v_\varphi \varphi) + (c^2 + c_\varphi \varphi)^2(u_\varphi - \varphi)(v_\varphi - \varphi)}
\]

**Generalized Multiplication**

Multiplication of \( a_\varphi \) by \( \alpha \) will be represented by

\[
 \alpha \otimes_\varphi a_\varphi
\]

It should fulfill the following requirements

\[
 1 \otimes_\varphi a_\varphi = a_\varphi
\]

\[
 0 \otimes_\varphi a_\varphi = \varphi
\]

\[
 (-\alpha) \otimes_\varphi a_\varphi = \alpha \otimes_\varphi N(a_\varphi)
\]

\[
 \infty \otimes_\varphi a_\varphi = c_\varphi
\]

Consistency with addition (41) requires

\[
 \left( \alpha \otimes_\varphi u_\varphi \right) \oplus_\varphi \left( \alpha \otimes_\varphi v_\varphi \right) = \alpha \otimes_\varphi \left( u_\varphi \oplus_\varphi v_\varphi \right)
\]

\[
 \left( \alpha \otimes_\varphi u_\varphi \right) \oplus_\varphi \left( \beta \otimes_\varphi u_\varphi \right) = (\alpha + \beta) \otimes_\varphi u_\varphi
\]

All our requirements are fulfilled if

\[
 \alpha \otimes_\varphi a_\varphi = \frac{M_0 + \varphi}{1 - M_0 \varphi / c^2}
\]
where

\[
M_{\alpha} = \left( \frac{c_{\varphi} - \varphi}{1 + c_{\varphi} \varphi / c^2} \right) \left( 1 + \frac{a_{\varphi} - \varphi}{1 + a_{\varphi} \varphi / c^2} \left( \frac{1 + c_{\varphi} \varphi / c^2}{c_{\varphi} - \varphi} \right)^{\alpha} \right) - \left( \frac{a_{\varphi} - \varphi}{1 + a_{\varphi} \varphi / c^2} \left( \frac{1 + c_{\varphi} \varphi / c^2}{c_{\varphi} - \varphi} \right)^{\alpha} \right) \right)
\]

(51)

**Discreteness**

**Discreteness Theorem:**

\[
\alpha \otimes_{\varphi} R(a_{\varphi}) = \begin{cases} 
\alpha \otimes_{\varphi} a_{\varphi} & \text{if } \alpha = \text{even integer} \\
R(\alpha \otimes_{\varphi} a_{\varphi}) & \text{if } \alpha = \text{odd integer}
\end{cases}
\]

(52)

**Proof:** Using (30), (49), (34) and (36), one can prove (52)

**Conclusion**

We have been able to develop a reciprocal symmetric algebra. We have also shown
the isomorphism between Einsteinian, de Broglie and quantum sets of numbers.

**References**