Derivation of Non-Einsteinian Relativistic Equations from Momentum Conservation Law

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Abstract

In this paper, we present a relativistic expression for the momentum conservation law using a formalism derived from the Lorentz-Einstein law for the addition of velocities. It demonstrates that a mass-less particle can move only at the speed of light and it has momentum like a photon. Hence, consideration of relativistic momentum conservation of a photon-gun system reveals that the mass of the gun and the time period, amplitude and the speed of photon vary with the speed of the gun. That is the mass, time, length and velocity are relative quantities. However, the corresponding expressions are different from those in Einstein’s theory of special relativity. Hence, we call those as Non-Einsteinian relativistic expressions. Moreover, each relative quantity has been found to have two values – one in the longitudinal and the other in the transverse directions, such that the product of the two remains invariant. Nevertheless, the longitudinal and transverse values have been denoted by the suffixes NL and NT respectively. It should be pointed here that, in the previous papers, the corresponding expressions in Einsteinian relativistic case have been denoted by the suffixes EL and ET respectively.

Keywords: Momentum conservation, Lorentz – Einstein law, Non-Einsteinian relativity, photon, relative mass, time, length and velocity.

INTRODUCTION

As a way of illustrating the classical momentum conservation law, let us take the example of a gun. Suppose, the mass of the gun is \( m \) and that of a bullet is \( m' \). When the bullet is fired, it moves with a velocity \( V \) in the forward direction. As a consequence, the gun moves in the backward direction with a velocity \( v \). Then, according to Newton’s third law of motion or the momentum conservation law\(^ {1} \)

\[
m'V = mv
\]

where, \( m'V \) is the momentum of the bullet and \( mv \) is that of the gun.

The relativistic multiplication of a velocity \( u \) by any number \( N \) is given by\(^ {2-6} \)

\[
N \otimes u = c \frac{\left( \frac{1+u/c}{1-u/c} \right)^N - 1}{\left( \frac{1+u/c}{1-u/c} \right)^N + 1}
\]

where, the symbol \( \otimes \) indicates the relativistic multiplication according to Lorentz - Einstein (L – E) law\(^ {7-9} \) and \( c \) is the speed of light.
On the other hand, according to Einstein’s special theory of relativity, the relative mass \( m \) of a particle of rest mass \( m_0 \) moving at a velocity \( v \) is given by

\[
m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

(3)

where, \( c \) is the speed of light. In this paper, we present a relativistic expression for the momentum conservation law making use of Eq. (2) in Eq. (1). It has been demonstrated that classically a mass-less particle will have an infinite speed, whereas, relativistically it can move only at the speed of light. Its overall behavior has been characterized as that of a photon. Hence, we have studied the momentum conservation of a photon emitted from a gun called photon-gun. In order to have better understanding of the relativistic effects, Non-Einsteinian relativistic case has been introduced to describe the variation of relative quantities like mass, time, length and the speed of photon with velocity of the gun. For each relative quantity, we have also considered two values - longitudinal and transverse.

Further, Einstein proposed a particle or photon model, which is also known as the quantum model, of light. In that model, he viewed light as consisting of streams of particles, called photons, rather than of wave. The energy content of each photon is equal to the product of Plank’s constant and the frequency of light. That is

\[
E = h\nu
\]

(4)

where,

- \( E \) = energy of photon
- \( h \) = Plank’s constant
- \( \nu \) = frequency of light

It should be pointed here that the rest mass of a photon is zero. However, it has momentum which can be obtained from the relation

\[
p = \frac{h\nu}{c}
\]

(5)

and the equivalent mass \( m' \) of a photon is given by

\[
m' = \frac{h\nu}{c^2}
\]

(6)

The energy of a photon can be expressed in terms of its momentum as

\[
E = pc
\]

(7)

where, \( p \) is the momentum of the photon and \( c \) is the speed of light. In the study, presented here, we have used the above mentioned photon model of light to describe the kinematics of a photon-gun system emitting a photon. As a consequence, we have
found the expressions for relative mass, time, length and velocity in the Non-Einsteinian relativistic case.

**Relativistic Description of Momentum Conservation Law**

From the momentum conservation law as given by Eq. (1), we can write

\[ V = \left( \frac{m}{m'} \right) v \]  

(8)

However, Eq. (8) does not limit the value of \( V \) within that of the speed of light \( c \) as required by the principle of relativity. For example, if we put \( m' \rightarrow 0 \) (mass-less particle) in the above equation, \( V \) becomes infinite (i.e. \( V \rightarrow \infty \)). It should be pointed here that in classical mechanics, the speed of light is considered to be infinite. But we can ensure \( V \leq c \) using the formalism given by Eq. (2) in Eq. (8) as demonstrated below. Let us express the above Equation as

\[ m' \otimes V = m \otimes v \]  

(9)

Multiplying both sides by \( 1/m' \), we get

\[ \frac{1}{m'} \otimes m' \otimes V = \frac{1}{m'} \otimes m \otimes v \]  

(10)

According to the properties of the formalism described in a previous paper, the above equation takes the form

\[ V = \left( \frac{m}{m'} \right) \otimes v \]  

(11)

Then, since \( m/m' \) is a number, the above equation can be written, following Eq. (2), as

\[ V = c \left( \frac{1 + v/c}{1 - v/c} \right)^{m/m'} - 1 \]  

\[ \left( \frac{1 + v/c}{1 - v/c} \right)^{m/m'} + 1 \]  

(12)

The above equation shows that the value of \( V \) cannot exceed \( c \). Further, it can be shown that in the classical limit, \( v \ll c \), it reduces to the classical momentum conservation law given by Eq. (1) as follows:

Equation (12) can also be written as
If $v \ll c$, then

$$V = c \left( \frac{m + v}{m'} - \frac{m'}{m} \right) $$

which is the classical momentum conservation law represented by Eq. (8). Hence, we can conclude that Eq. (12) represents the relativistic expression for the momentum conservation law. The equation also indicates that $V \to c$ if $m/m' \to \infty$. One possibility is when $m' \to 0$ (mass-less particle). It means that a mass less particle moves at the speed of light $c$. Further, from Eq. (12), $V = 0$ for $v = 0$, whereas, $V = c$ for any non zero value of $v$ (i.e. $v \neq 0$) and if $m' \to 0$ (i.e. $m/m' \to \infty$). These results indicate the following:

A mass less particle does not move unless some momentum is imparted to it. Therefore, according to the momentum conservation law, its momentum must be equal to the recoil momentum of the source. However, its momentum or change of momentum cannot be accounted for by its mass (0) and velocity (c). These are the characteristics possessed by a photon according to Einstein’s model of light.

**Implications of the Results**

(i) **Relative mass**

In the above discussion, the mass of the bullet was considered to be negligible compared to that of the gun. If the mass of the bullet is taken into consideration, the momentum conservation law becomes

$$m'V = (m_0 - m')v$$

where, $m_0$ is the total mass of the gun and bullet at rest and $m'$ is the mass of the bullet. Now, if a photon having energy $\hbar v$ is emitted from the gun, since the rest mass of a photon is zero, we can write from the energy conservation law

$$\hbar v = \frac{1}{2} m_0 v^2$$

The term on the RHS of the above equation represents the kinetic energy of the gun. Further, since a photon has momentum, the momentum conservation law given by Eq. (15), can be written using Eqs. (5) and (6) as

$$\frac{\hbar v}{c} = \left( \frac{m_0 - \hbar v}{c^2} \right) v$$
or

\[ \frac{h \nu}{c} = m_0 v - \left( \frac{h \nu}{c} \right) \left( \frac{v}{c} \right) \]  \hspace{1cm} (18)

The second term on the RHS is \((v/c)\) times the momentum of the photon. Using Eq. (16) in Eq. (17), we get

\[ \frac{h \nu}{c} = \left( m_0 - \frac{1}{2} m_0 \frac{v^2}{c^2} \right) v \]

\[ = m_0 \left( 1 - \frac{1}{2} \frac{v^2}{c^2} \right) v \]  \hspace{1cm} (19)

\[ = m_0 \left( 1 - \frac{1}{2} \frac{v^2}{c^2} \right) v \]  \hspace{1cm} (20)

The above equation indicates that the mass of the gun decreases with velocity. Hence, we can infer that the mass is relative. Now, in the Einsteinian relativistic case, all the relative quantities were found to be decreasing with velocity in the longitudinal direction. Hence, in the above case, the variation of mass with velocity can be regarded as the relative mass in the longitudinal direction. However, it is different from the corresponding expression in Einsteinian relativity. Hence, we call this case as Non-Einsteinian relativity. So, let us denote the relative mass, in this case, by \( m_{NL} \). Where, in the suffix, \( N \) stands for Non-Einsteinian and \( L \) for longitudinal. Henceforth, we will use the same suffix for all relative quantities in the longitudinal direction and the suffix \( NT \) for transverse relative values for them. Therefore, the RHS of the above equation can be expressed as

\[ m_{NL} = m_0 \left( 1 - \frac{1}{2} \frac{v^2}{c^2} \right) \]  \hspace{1cm} (21)

Now, multiplying both sides of the above equation by \( c^2 \) and rearranging terms, we get

\[ m_0 c^2 = m_{NL} c^2 + \frac{1}{2} m_0 v^2 \]  \hspace{1cm} (22)

which is the energy conservation law. This means that the total energy remains constant at the rest mass energy. Further, dividing both sides of the above equation by \( \left( 1 - \frac{1}{2} \frac{v^2}{c^2} \right) \),

\[ \frac{m_0 c^2}{\left( 1 - \frac{1}{2} \frac{v^2}{c^2} \right)} = m_0 c^2 + \frac{1}{2} \frac{m_0 v^2}{\left( 1 - \frac{1}{2} \frac{v^2}{c^2} \right)} \]  \hspace{1cm} (23)

or
\[ m_{NT}c^2 = m_0c^2 + \frac{1}{2} m_{NT}v^2 \]  
\[ \text{(24)} \]

where,

\[ m_{NT} = \frac{m_0}{\left(1 - \frac{v^2}{2c^2}\right)} \]  
\[ \text{(25)} \]

is the transverse mass in the Non-Einsteinian relativistic case. It is clear that \( m_{NT} \) increases with increasing velocity and hence it is comparable with relative mass given by Eq. (3) in Einsteinian relativity. However, the second term on the RHS of Eq. (24) represents the relativistic kinetic energy. Therefore, from Eqs. (21) and (25), we get

\[ m_{NL}m_{NT} = m_0^2 \]  
\[ \text{(26)} \]

The above equation indicates that the product of the longitudinal and transverse masses is equal to the square of the rest mass. It means that the rest mass is an invariant quantity and is equal to the geometric mean of the longitudinal and transverse masses.

\textbf{(II) Relative Momentum and Energy of the Gun}

According to Einstein’s mass-energy equivalence principle, the rest mass energy \( (E_0) \) is given by:

\[ E_0 = m_0c^2 \]  
\[ \text{(27)} \]

\[ \therefore \quad m_0 = E_0/c^2 \]  
\[ \text{(28)} \]

Using Eq. (28) in Eq. (26), we obtain

\[ (m_{NL}c^2)(m_{NT}c^2) = E_0^2 \]  
\[ \text{(29)} \]

or

\[ E_{NL}E_{NT} = E_0^2 \]  
\[ \text{(30)} \]

where,

\[ E_{NL} = m_{NL}c^2 = E_0\left(1 - \frac{1}{2} \frac{v^2}{c^2}\right) \]  
\[ \text{(31)} \]

is the relative energy in the longitudinal direction and

\[ E_{NT} = m_{NT}c^2 = \frac{E_0}{\left(1 - \frac{1}{2} \frac{v^2}{c^2}\right)} \]  
\[ \text{(32)} \]
is the relative energy in the transverse direction. Hence, from Eq. (30), we can conclude that the product of the relative energies in the longitudinal and transverse directions is equal to the square of the rest mass energy. It means that the rest mass energy is an invariant quantity and is equal to the geometric mean of the longitudinal and transverse energies. Equation (29) can also be written as

$$(m_{NL}c)(m_{NT}c) = (E_0/c)^2$$

(33)

$$p_{NL}p_{NT} = p_0^2$$

(34)

where,

$$p_0 = E_0/c = m_0c$$

(35)

is the momentum related to the rest mass energy or Compton momentum,

$$p_{NL} = m_{NL}c = p_0 \left(1 - \frac{1}{2} \frac{v^2}{c^2}\right)$$

(36)

is the relative momentum in the longitudinal direction and

$$p_{NT} = m_{NT}c = \frac{p_0}{\left(1 - \frac{1}{2} \frac{v^2}{c^2}\right)}$$

(37)

is the relative momentum in the transverse direction. Hence, from Eq. (34), we can conclude that the product of the momentums in the longitudinal and transverse directions is equal to the square of the Compton momentum. It means that the Compton momentum is an invariant quantity and is equal to the geometric mean of the longitudinal and transverse momentums.

**(iii) Relative Velocity of the Photon**

Equation (33) can also be expressed as

$$(m_0v_{NL})(m_0v_{NT}) = m_0^2c^2$$

(38)

$$v_{NL}v_{NT} = c^2$$

(39)

where,

$$v_{NL} = c \left(1 - \frac{1}{2} \frac{v^2}{c^2}\right)$$

(40)

is the relative velocity in the longitudinal direction and

$$v_{NT} = \frac{c}{\left(1 - \frac{1}{2} \frac{v^2}{c^2}\right)}$$

(41)
is the relative velocity in the transverse direction. Hence, from Eq. (39), we can conclude that the product of the relative velocities in the longitudinal and transverse directions is equal to the square of the velocity of the photon or the speed of light. It means that the speed of light is an invariant quantity and is equal to the geometric mean of the longitudinal and transverse velocities.

(IV) RELATIVE TIME

Putting \( c = \lambda_0 / t_0 \) (\( \lambda_0 \) is the wavelength and \( t_0 \) is the time period of the photon) in Eq. (39), we obtain

\[
\frac{\lambda_0}{t_0} \left( 1 - \frac{1}{2} \frac{v^2}{c^2} \right) \frac{\lambda_0}{t_0} \left( 1 - \frac{1}{2} \frac{v^2}{c^2} \right) = \frac{\lambda_0^2}{t_0^2}
\]

or

\[
\frac{\lambda_0^2}{t_{NL} t_{NT}} = \frac{\lambda_0^2}{t_0^2}
\]

\[
\therefore \quad t_{NL} t_{NT} = t_0^2
\]

It should be pointed here that the time period \( t_0 \), generally, indicates an interval of time which is the notion of time in modern physics. In other words, the time period can correctly represent the discrete nature of time which is consistent with the concept of relativity. Therefore, in Eq. (44),

\[
t_{NL} = t_0 \left( 1 - \frac{1}{2} \frac{v^2}{c^2} \right)
\]

represents the relative time in the longitudinal direction which is contracted and

\[
t_{NT} = \frac{t_0}{\left( 1 - \frac{1}{2} \frac{v^2}{c^2} \right)}
\]

represents the relative time in the transverse direction which is dilated. Thus, we can conclude that time is contracted in the longitudinal direction but dilated in the transverse direction. Moreover, from Eq. (44), we can conclude that \( t_0 \) is an invariant quantity and is equal to the geometric mean of the longitudinal and transverse times.

(V) RELATIVE LENGTH

Since, \( \lambda_0 \) is an interval between two space points, it can be considered as a correct representation of length \( (l_0) \) in relativistic case. Hence, Eq. (42) can be written as
\[
\frac{l_0}{t_0} \left(1 - \frac{1}{2} \frac{v^2}{c^2}\right) = \frac{l_0^2}{t_0^2} \tag{47}
\]

or

\[
\frac{l_{NL}l_{NT}}{t_0^2} = \frac{l_0^2}{t_0^2} \tag{48}
\]

\[
\therefore \quad l_{NL}l_{NT} = l_0^2 \tag{49}
\]

where,

\[
l_{NL} = l_0 \left(1 - \frac{1}{2} \frac{v^2}{c^2}\right) \tag{50}
\]

represents the relative length in the longitudinal direction which is contracted and

\[
l_{NT} = \frac{l_0}{\left(1 - \frac{1}{2} \frac{v^2}{c^2}\right)} \tag{51}
\]

represents the relative length in the transverse direction which is dilated. Thus, we can conclude that length is contracted in the longitudinal direction but dilated in the transverse direction. Moreover, from Eq. (49), we can conclude that \(l_0\) is an invariant quantity and is equal to the geometric mean of the longitudinal and transverse lengths.

From the above discussions we can conclude that, as a general rule, any relative quantity will have longitudinal and transverse values such that their product will remain invariant at the square of its value at rest. Symbolically, it can be written as

\[
X_{NL}X_{NT} = X_0^2 \tag{52}
\]

where, \(X\) is any relative quantity and \(X_0\) is its value when \(v \to 0\). This is valid for Eq. (39) as well because it is clear from Eqs. (40) and (41) that \(v_{NL} = v_{NT} = c\) when \(v \to 0\).

Further, it is obvious from the above results that all relative quantities decrease in the longitudinal direction but increase in the transverse direction with increasing velocity.

We summarize the results obtained in the following table. The table contains a complete set of self consistent equations for the relative mass, time, length, velocity, momentum and energy in the case of Non-Einsteinian view of relativity. For each of the relative quantities, both the longitudinal and transverse values and the value of their product are given.
Table 1: Relative quantities in Non-Einsteinian view of relativity

<table>
<thead>
<tr>
<th>Relative Quantity</th>
<th>Transverse</th>
<th>Longitudinal</th>
<th>Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass</td>
<td>$m_{NT} = \frac{m_0}{\left(1 - \frac{v^2}{2c^2}\right)}$</td>
<td>$m_{NL} = m_0\left(1 - \frac{v^2}{2c^2}\right)$</td>
<td>$m_{NL}m_{NT} = m_0^2$</td>
</tr>
<tr>
<td>Time</td>
<td>$t_{NT} = \frac{t_0}{\left(1 - \frac{v^2}{2c^2}\right)}$</td>
<td>$t_{NL} = t_0\left(1 - \frac{v^2}{2c^2}\right)$</td>
<td>$t_{NL}t_{NT} = t_0^2$</td>
</tr>
<tr>
<td>Length</td>
<td>$l_{NT} = \frac{l_0}{\left(1 - \frac{v^2}{2c^2}\right)}$</td>
<td>$l_{NL} = l_0\left(1 - \frac{v^2}{2c^2}\right)$</td>
<td>$l_{NL}l_{NT} = l_0^2$</td>
</tr>
<tr>
<td>Velocity</td>
<td>$v_{NT} = \frac{c}{\left(1 - \frac{v^2}{2c^2}\right)}$</td>
<td>$v_{NL} = c\left(1 - \frac{v^2}{2c^2}\right)$</td>
<td>$v_{NL}v_{NT} = c^2$</td>
</tr>
<tr>
<td>Momentum</td>
<td>$p_{NT} = \frac{p_0}{\left(1 - \frac{v^2}{2c^2}\right)}$</td>
<td>$p_{NL} = p_0\left(1 - \frac{v^2}{2c^2}\right)$</td>
<td>$p_{NL}p_{NT} = (p_0)^2$</td>
</tr>
<tr>
<td>Energy</td>
<td>$E_{NT} = \frac{E_0}{\left(1 - \frac{v^2}{2c^2}\right)}$</td>
<td>$E_{NL} = E_0\left(1 - \frac{v^2}{2c^2}\right)$</td>
<td>$E_{NL}E_{NT} = E_0^2$</td>
</tr>
</tbody>
</table>

The relativistic Kinetic Energy (K.E.), from Eq. (24), is

$$K.E. = \frac{1}{2}m_{NT}v^2 = m_{NT}c^2 - m_0c^2$$ \hspace{1cm} (53)

**Conclusions**

Through using the relativistic multiplication rule for the velocities we have found:

(a) Relativistic expression for the momentum conservation law.
(b) A mass less particle can move only at the speed of light and has momentum.
(c) Conservation of momentum for a photon-gun system reveals (as implications):
   (i) A complete set of self consistent equations of relative quantities in non-Einsteinian relativistic case. In this case there, is no restriction on any particle’s speed being equal to or greater than that of light.
(ii) The relative mass increases in the transverse direction but decreases in the longitudinal direction. However, the rest mass is an invariant quantity.

(iii) The relative length is contracted in the longitudinal direction but dilated in the transverse direction. However, the length at rest is an invariant quantity.

(iv) The relative time is dilated in the transverse direction but contracted in the longitudinal direction. However, the time at rest is an invariant quantity.

(v) The relative velocity decreases in the longitudinal direction but increases in the transverse direction. However, the product of the longitudinal and transverse velocities is equal to the square of the speed of light. It means the speed of light is an invariant quantity and is equal to the geometric mean of the two relative velocities.

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