Incompatibility between Einstein's Postulate and Special Theory of Relativity

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Abstract

A hyperbolic transformation relates relativistic velocities to corresponding Galilean velocity or rapidity. Comparison shows that usual definition of velocity is valid in Euclidian space but not valid in relativistic hyperbolic space. It also reveals that Einstein's postulate -- that the speed of light is the same in all inertial frames of reference -- is equivalent to the trivial statement that infinite velocity is the same in all inertial frames of reference.

Key Words: Galilean; Relativistic; Hyperbolic transformation; Lorentz transformation; Einstein’s postulates; Einstein’s addition; Velocity.
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INTRODUCTION

An isomorphic transformation relates Galilean velocities (also called rapidity¹) to corresponding (relativistic²) velocities. We intend to explore this transformation and study, in particular, how \( c \) the speed of light transforms. We hope this study will give us an insight into the meaning of “velocity” and Einstein's postulate. We shall also study relativistic kinematics in Galilean algebra and Galilean kinematics in relativistic algebra. This will also give us further insight into relativistic kinematics and make it possible to examine to what extent Special Relativity fulfills Einstein’s postulate.

HYPERBOLIC TRANSFORMATION AND RELATIVISTIC VELOCITY COMPOSITION

Rapidity\(^1\) \( U \) corresponding to velocity \( u \) is defined by\(^3\)

\[
U/c = \text{arctanh}(u/c) \quad \text{or} \quad u/c = \tanh(U/c) \quad \text{with} \quad -1 \leq u/c \leq 1
\]

\( c \) is the speed of light. (1) permits us to write the following identity\(^4\)

\[
U/c = \ln \psi(u/c) = \ln \frac{1+u/c}{\sqrt{1-(u/c)^2}}
\]

The Galilean relative velocity \( W = U - V \) becomes, using (2)

\[
W/c = U/c - V/c = \ln \psi(u/c) \psi(-v/c) = \ln(w/c)
\]

Where the corresponding relativistic relative velocity is
\[
w = u \oplus (-v) = \frac{u - v}{1 - u \cdot v / c^2}
\] with \(-1 \leq w / c \leq 1\) \hspace{1cm} (4)

**Transformation of \(c\)**

Case 1: \(u = u_c\) corresponds to \(U = c\). (1) gives

\[
u_c / c = \tanh(c/c) = \tanh 1 = \frac{e^2 - 1}{e^2 + 1}
\] \hspace{1cm} (5)

Case 2: \(u = c\) corresponds to \(U = \infty\). (1) gives

\[
c / c = 1 = \tanh(\infty / c)
\] \hspace{1cm} (6)

In Case 1, \(c\) is not invariant under Galilean addition and its analogue \(u_c\) is not invariant under Einsteinian addition. This contradicts Einstein’s postulate.

In Case 2, \(c\) is invariant under Einsteinian addition (4) and its analogue \(\infty\) is invariant under Galilean addition. \(\infty - V = \infty\) is a trivial relation, which makes Einstein’s postulate trivial.

The question now is, which of the above two cases (Case 1 or Case 2) is the correct choice. We now have to see what we mean by ‘velocity’.

**Law of Multiplication and Definition of Velocity**

From (2)

\[
nU / c = n \cdot \ln \psi(u / c) = \ln(\psi(u / c))^n = \ln \psi((n \otimes u) / c)
\] \hspace{1cm} (7)

Where the relativistic law of multiplication is given by

\[
n \otimes u = c \left(\frac{1 + u / c}{1 - u / c}\right)^n - 1
\] \hspace{1cm} (8)

Multiplication \(nU\) is consistent with Galilean law of addition in the sense that it is distributive with this addition. Similarly \(n \otimes u\) of (8) is consistent with Einsteinian law of addition in the sense that it is distributive with addition (4).

\[
\{m \otimes u\} \oplus \{m \otimes v\} = m \otimes \{u \oplus v\} \quad \text{and}
\]

\[
\{m \otimes u\} \oplus \{n \otimes u\} = (m + n) \otimes u
\] \hspace{1cm} (9)
To find the correct definition for velocity we must find the transformation for distance corresponding to (1). We chose a constant time $T$ and define an upper bound distance $L = cT$ corresponding to upper bound velocity $c$. We write corresponding to (1)

$$X/cT = \text{arctanh}(x/cT)$$

or

$$x/cT = \tanh(X/cT)$$

(10)

Then we can write, using (2) and (10)

$$U/c = (X/t)/c = (X/cT)(T/t) = (T/t) \ln \psi(x/cT) = \ln(\psi)^{T/t}$$

$$= \ln \psi([T/t]) \otimes (x/cT)] = \ln \psi(u/c)$$

(11)

Where

$$u = c(T/t) \otimes (x/cT) = c \frac{1 + x/cT}{1 - x/cT} - 1$$

(12)

Therefore, for the relativistic definition of velocity we have to replace $X/t$ by $(T/t) \otimes (x/cT)$. Using (10), (12) may be written as

$$u = c(T/t) \otimes (x/cT) = c \frac{e^{2(X/cT)(T/t)} - 1}{e^{2(X/cT)(T/t)} + 1} = c \frac{e^{2(X/t)/c} - 1}{e^{2(X/t)/c} + 1}$$

(13)

In the limit $cT \to \infty$ (13) gives

$$u = c(T/t) \otimes (x/cT) \xrightarrow{cT \to \infty} X/t$$

(14)

Galilean speed of light corresponds to $X/t = c$. At this value (13) gives $u = u_c$ of (5) above. When $t = 0$ corresponding to $X/t = \infty$, (13) gives $u = c$.

$$X/t \xrightarrow{t \to 0} \infty \text{ and } u = c(T/t) \otimes (x/cT) \xrightarrow{t \to 0} c$$

(15)

**Galilean Physics in Relativistic Space**

The usual way of distinguishing between Galilean and relativistic kinematics is by their laws of addition (3) and (4), keeping the same definition of velocity (the same law of multiplication). An alternative way of distinguishing between them is by their definitions of velocity keeping the same (Einsteinian) law of composition of velocities. If the composition of velocities is given by (4) and velocity is defined in the usual Galilean way as in $u = x/t$, we are in special relativity (section 6 below). On the other hand, keeping the same law of composition (4), if velocity is defined by
\[ u = \left( \frac{T}{t} \right) \otimes \left( \frac{x}{T} \right) \] of (12), we are describing Galilean physics in relativistic algebra i.e. in hyperbolic space instead of the usual Euclidian space.

**Relativistic Physics in Euclidian Space**

In section 5 we have presented Galilean physics in hyperbolic space. In this section we want to present relativistic physics in Galilean algebra. We shall keep Galilean law of addition of velocities, but modify the definition of velocity. If a body covers (hyperbolic-space) distance \( x \) in time \( t \), we shall define two velocities, Galilean velocity \( \dot{x} = x/t \) and relativistic velocity \( \ddot{x} \) (in Galilean space) defined below

\[
\ddot{x} = \frac{c}{2} \ln \left( \frac{1 + \dot{x}/c}{1 - \dot{x}/c} \right) \text{ with } -1 \leq \dot{x}/c \leq 1 \text{ and } \dot{x} = x/t
\] (16)

If two relativistic velocities are added in the usual Galilean way

\[
\ddot{x} + \ddot{y} = \frac{c}{2} \ln \left( \frac{1 + (\dot{x} \oplus \dot{y})/c}{1 - (\dot{x} \oplus \dot{y})/c} \right) = c \ln \frac{1 + (\dot{x} \oplus \dot{y})/c}{\sqrt{1 - (\dot{x} \oplus \dot{y})^2 / c^2}}
\] (17)

Where \( \dot{x} \oplus \dot{y} \) is given by (4). Therefore, Galilean addition of relativistic velocities (left hand side of (17)) corresponding to relativistic addition of usual velocities. Therefore, (17) represents relativistic physics although the addition is Galilean\(^3\). (16) shows

\[ -\infty \leq \ddot{x} \leq \infty \text{ as } -c \leq \dot{x} \leq c \] (18)

(18) agrees with (6). Defining constant time \( T \) and constant length \( L \), we can define \( \ddot{x} = V_{RG}(X,t) \) in terms of \( X \) and \( t \) as below.

\[
\ddot{x} = V_{RG}(X,t) = \frac{c}{2} \ln \left( \frac{1 + (T/t) e^{2x/L}}{e^{2x/L} + 1} \right)
\] (19)

**Entanglement of Spaces**

In Special Relativity one employs the Galilean definition of velocity \( (x/t) \), but employs hyperbolic space law of addition (4). There is an inconsistent mixture of spaces\(^7\), which leads to confusion.

**Triviality of Einstein’s Postulate**

Einstein’s postulate: The speed of light is the same in every inertial frame of reference. When \( u = c \) corresponding to Case 2 of section 3, \( c \) is invariant under Einstein’s addition (4). Therefore, postulate 1 above is fulfilled, but it corresponds to Galilean relation \( + v = \infty \) (15), which is trivial. Triviality apart, Special Relativity is also internally inconsistent because multiplication (and the definition of velocity) is not
distributive with the addition, as we have seen in sections 4, 5 and 6. Galilean kinematics is consistent both in Euclidean space and in hyperbolic space because laws of addition and multiplication are consistent; multiplication is distributive with the addition.

**Logical Incongruity in Einstein’s Postulate**

Einstein’s postulate was conceived in Galilean space. The postulate is

\[ c - V = c \]  \hspace{1cm} (20)

where \( c \) is the (finite, Galilean) speed of light and \( V \neq 0 \). (20) is never true. Therefore, for finite \( c \) and \( V \neq 0 \) (20) is an absurdity. When one tried to impose a relation which it does not accept mathematically, one (unknowingly) pushed it into a hyperbolic space, which accepts relation (4),

\[ c \oplus (-V) = c \]  \hspace{1cm} (21)

but failed to notice that this shift of space demands that the velocities be consistently transformed. This has not been done. In recognition of the mathematical inconsistency, one calls the rule (4) the “composition rule” of velocities, and not the “law of addition” of velocities. By “composition” is meant that it is an heuristic rule not supported by mathematical discipline.

In the pursuit of sciences, we apply logic and mathematics. This is based on the belief (probably the most fundamental belief in the study of sciences) that the manifest universe, which contains this physical world, is harmonious with reason and conforms to mathematics. A postulate is something we accept without proof, but it has to have the plausibility of acceptance. It should not be a mathematical absurdity.

**Addition of Distance in Hyperbolic Space**

Using (2) and (10) and setting \( L = cT \)

\[ Y/L + X/L = \ln \psi(y/L)\psi(x/L) = \left( \frac{(y + x)/L}{1 + y.x/(L^2)} \right) \]  \hspace{1cm} (22)

\( x \) and \( y \) obey Einstein’s law of addition of velocities with \( c \) replaced by \( L \). In (22) \( L \) is an invariant under addition and determines the scale just as \( c \) is an invariant under Einsteinian addition and determines the velocity scale. (22) is an important deviation from Special Relativity. In Special Relativity Einstein’s law of addition is valid for velocities only.

**Conclusion**

Special Theory of Relativity was developed to translate Einstein’s postulate. The postulate is internally inconsistent, and leads to an inconsistent mixture of Euclidian
and hyperbolic spaces. Either it is not true as in Case 1 of section 3 above, or it is true but trivial as in Case 2.

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This mixing has also been observed by Saleh H. Naqib. Private communication.